The Twin Paradox
[several articles that discuss this thought experiment about time]

How does relativity theory resolve the Twin Paradox?

Ronald C. Lasky, a lecturer at Dartmouth College's Thayer School of Engineering, explains.

Time must never be thought of as pre-existing in any sense; it is a manufactured quantity. -- Hermann Bondi

Paul Davies's recent article "How to Build a Time Machine" has rekindled interest in the Twin Paradox, arguably the most famous thought experiment in relativity theory. In this supposed paradox, one of two twins travels at near the speed of light to a distant star and returns to the earth. Relativity dictates that when he comes back, he is younger than his identical twin brother.

The paradox lies in the question "Why is the traveling brother younger" Special relativity tells us that an observed clock, traveling at a high speed past an observer, appears to run more slowly. (Many of us solved this problem in sophomore physics, to demonstrate one effect of the absolute nature of the speed of light.) Since relativity says that there is no absolute motion, wouldn’t the brother traveling to the star also see his brother’s clock on the earth move more slowly? If this were the case, wouldn’t they both be the same age? This paradox is discussed in many books but solved in very few. When the paradox is addressed, it is usually done so only briefly, by saying that the one who feels the acceleration is the one who is younger at the end of the trip. Hence, the brother who travels to the star is younger. While the result is correct, the explanation is misleading. Because of these types of incomplete explanations, to many partially informed people, the accelerations appear to be the issue. Therefore, it is believed that the general theory of relativity is required to explain the paradox. Of course, this conclusion is based on yet another mistake, since we don't need general relativity to handle accelerations. The paradox can be unraveled by special relativity alone, and the accelerations incurred by the traveler are incidental. An explanation follows.

Let us assume that the two brothers, nicknamed the traveler and the homebody, live in Hanover, N.H. They differ in their wanderlust but share a common desire to build a spacecraft that can achieve 0.6 times the speed of light (0.6c). After working on the spacecraft for years, they are ready to launch it, manned by the traveler, toward a star six light-years away. His craft will quickly accelerate to 0.6c. For those who are interested, it would take a little more than 100 days to reach 0.6c at an acceleration of 2g's. Two g's is two times the acceleration of gravity, about what one experiences on a sharp loop on roller coaster. However, if the traveler were an electron, he could be accelerated to 0.6c in a tiny fraction of a second. Hence, the time to reach 0.6c is not central to the argument. The traveler uses the length-contraction equation of special relativity to measure distance. So the star six light-years away to the homebody appears to be only 4.8 light-years away to the traveler at a speed of 0.6c. Therefore, to the traveler, the trip to the star takes only eight years (4.8/0.6), whereas the homebody calculates it taking 10 years (6.0/0.6). It is instructive to discuss how each would view his and the other’s clocks during the trip. Let’s assume that each has a very powerful telescope that enables such observation. Surprisingly, with careful use of the time it takes light to travel between the two we can explain the paradox.

Both the traveler and homebody set their clocks at zero when the traveler leaves the earth for the star (event 1). When the traveler reaches the star (event 2) his clock reads eight years. However, when the homebody sees the traveler reach the star, the homebody’s clock reads 16 years. Why 16 years? Because, to the homebody, the craft takes 10 years to make it to the star and the light six additional years to come back to the earth showing the traveler at the star. So to the homebody, the traveler’s clock appears to be running at half the speed of his clock (8/16.)?

As the traveler reaches the star he reads his clock at eight years as mentioned, but he sees the homebody's clock as it was six years ago (the amount of time it takes for the light from the earth to reach him), or at four years (10-6). So the traveler also views the homebody's clock as running half the speed of his clock (4/8).

On the trip back, the homebody views the traveler's clock going from eight years to 16 years in only four years' time, since his clock was at 16 years when he saw the traveler leave the star and will be at 20 years when the traveler arrives back home (event 3). So the homebody now sees the traveler's clock advance eight years in four years of his time; it is now twice as fast as his clock. On the trip back, the traveler sees the homebody's clock advance from four to 20 years in eight years of his time. Therefore, he also sees his brother's clock advancing at twice the speed of his. They both agree, however, that at the end of the trip the traveler's clock reads 16 years and the homebody's 20 years. So the traveler is four years younger. The asymmetry in the paradox is that the traveler leaves the earth's reference frame and comes back, whereas the homebody never leaves the earth. It is also an asymmetry that the traveler and the homebody agree with the reading on the traveler's clock at each event, but not vice versa. The traveler's actions define the events.

The Doppler effect and relativity together explain this effect mathematically at any instant. The interested reader will find the combination of these effects discussed in *The Fundamentals of Physics,* by David Halliday et al. (John Wiley and Sons, 1996). Paul Davies also does a nice job explaining the Twin Paradox in his book *About Time* (Touchstone 1995, ppf 59.) My explanation follows Davies's closely; I hope my graph adds further clarity. The reader should also note that the speed that an observed clock appears to run depends on whether it is traveling away from or toward the observer. The sophomore physics problem, mentioned earlier, is a special case as it applies only when the motion of the traveler passes the observer's reference frame with no separating distance in the direction of motion.

For those with a little more formal physics background, a spacetime diagram also explains the paradox nicely. It is shown with the supporting calculations for the Doppler effect on the observed time. Proper time is time in the frame of the observer.?

- **The Setup**

The twin paradox is perhaps the most celebrated of the many possible paradoxes to consider in special relativity. The story goes something like this.

Suppose you have a set of twins. One of the twins stays on earth, but the other becomes an astronaut. The astronaut takes off and travels at a very high speed. According to time dilation, the clocks on the rocket ship tick slower than those on earth. He travels for a while, then turns around to return to earth. When he arrives he is younger than his twin. How can this be?

A Paradox on Two Levels

There are really two levels of weirdness to the story. The first level of weirdness is simply that the two twins are different ages. This is a basic consequence of time dilation. This is not a paradox, per se, just an instinctual reaction to the non-Newtonian way that time works. It doesn’t feel right for the two twins to have different biological ages, but that is relativity. Weird? Yes. Contradiction? No.

The second level of weirdness is more serious. The origin of the thought lies in the principle of relativity. Suppose we adopt the viewpoint of the traveling twin. He looks back on earth and it appears to be in motion — moving away at the same speed that he is moving relative to the earth. Why can’t he invoke the same time dilation logic? The answer is that he can. He must. He looks back at earth and it is the earth-bound twin that appears to be aging slower! When he returns, he will say that his twin is younger. Which will it be?

Now, this is a true paradox. They both can’t be younger than the other. The standard solution is to state that the traveling twin must turn around at some point in order to return to the earth. This acceleration invalidates his frame — it is not truly an inertial frame (one that moves at a constant speed in a constant direction). Therefore his conclusion is false. He is the younger one.

This solution is true, but only tells half of the story. What does the traveling twin actually see?

A Better Solution

Let’s suppose, for example, the traveling twin moves at a speed 60% the speed of light and travels 3.0 light-years (as measured from earth). It will take him 5.0 years to travel this distance — as measured from the earth. Let’s give a day or so for the turnaround, which we will neglect. The return trip is also at 60% the speed of light. So the total trip takes 10.0 earth years.

$$t' = \gamma t$$

$$L' = L/\gamma$$

$$\Delta t' = Lv/c^2$$

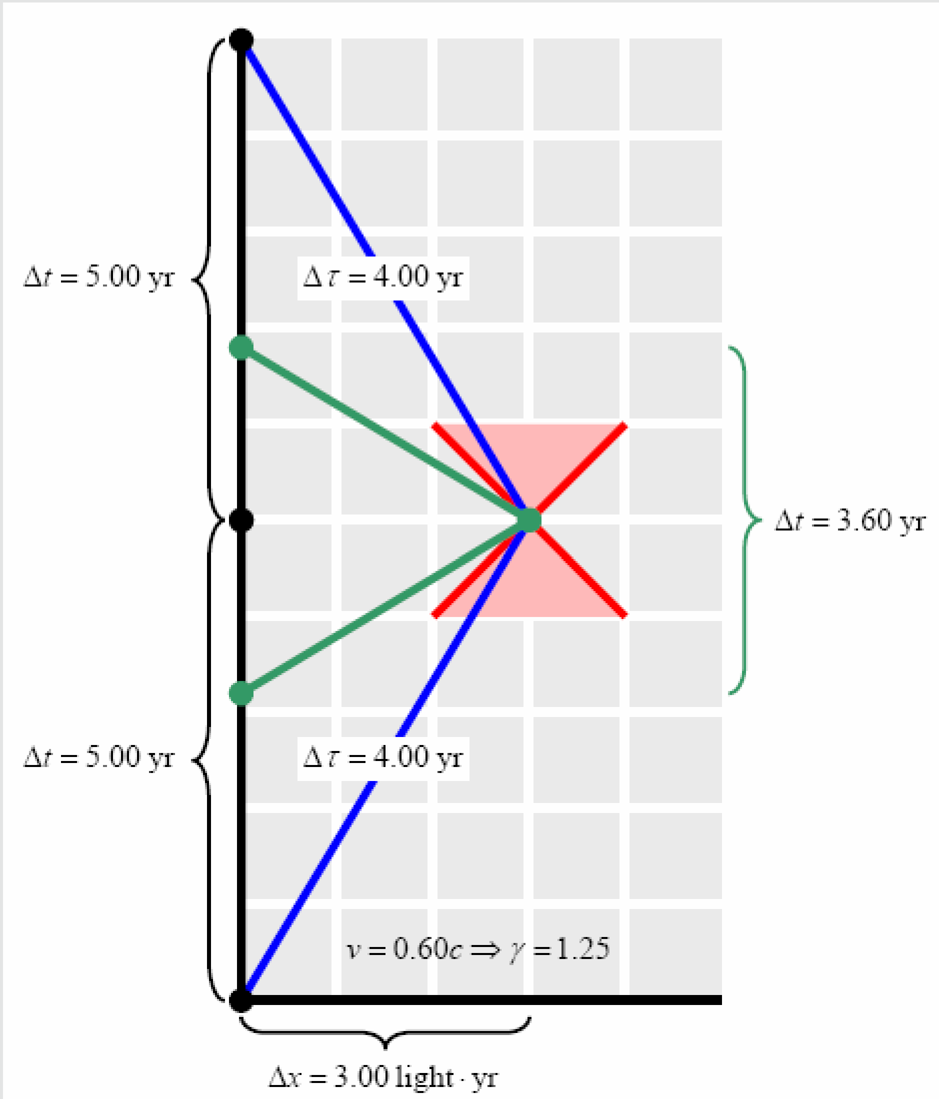
Three Relativistic Effects

Since the traveling twin moves, his clock ticks slower. At a speed of 60% the speed of light, the gamma factor is 1.25, so he only measures eight years on the trip. But according to the principle of relativity, the traveling twin may regard himself at rest. He sees the earthbound clock tick slow. In the eight years his clocks tick, he sees his twin age only 6.4 years.

He really does perceive only 6.4 years to pass on earth. But we have forgotten to take into account the desynchronization effect. Initially, the two clocks (biological in this case) are in sync with one another. But during his acceleration the clocks on board his ship go wildly out of sync with those on earth — 3.6 years out of sync to be precise.

This number comes from the fact that when he decelerates from 60% the speed of light to zero, there is a 3.0 light-year separation between him and his twin. The time de-sync factor is 1.8 years. Then, of course, he accelerates back up to 60% which introduces another 1.8 years of desynchronization into his clock. The perceived 6.4 years of earth-time plus the 3.6 years of desynchronization matches the 10.0 years of time seen on earth. Ultimately they both agree that 10.0 years pass on earth.

This is summarized in the following diagram.



Twin Paradox Explained

The red “X” represents the moment in time when the traveling twin turns around. Before he does, the bottom green line represents the events that he regards as simultaneous. When he turns around and heads back, his velocity is different and his line of simultaneity is also different. This is the top green line.

The bottom black dot to the bottom green dot are the events on earth that the traveling twin sees on his outbound trip. During his time of acceleration the events that occur between the two green dots appear to happen — 3.6 years in one day. On his trip back he sees the remaining black line above the top green dot.

So when the traveling twin returns he truly does see all ten years pass on earth. Paradox resolved.

The Twin Paradox: Introduction

Our story stars two twins, sometimes unimaginatively named A and B; we prefer the monikers Stella and Terence. Terence sits at home on Earth. Stella flies off in a spaceship at nearly the speed of light, turns around after a while, thrusters blazing, and returns. (So Terence is the terrestrial sort; Stella sets her sights on the stars.)

When our heroes meet again, what do they find? Did time slow down for Stella, making her years younger than her home-bound brother? Or can Stella declare that the *Earth* did the travelling, so Terence is the younger?

Not to keep anyone in suspense, Special Relativity (SR for short) plumps unequivocally for the first answer: Stella ages less than Terence between the departure and the reunion.

Perhaps we can make short work of the "travelling Earth" argument. SR does *not* declare that *all* frames of reference are equivalent, only so-called *inertial* frames. Stella's frame is not inertial while she is accelerating. And this is observationally detectable: Stella had to fire her thrusters midway through her trip; Terence did nothing of the sort. The Ming vase she had borrowed from Terence fell over and cracked. She struggled to maintain her balance, like the crew of Star Trek. In short, she *felt* the acceleration, while Terence felt nothing.

Whew! One short paragraph, and we've polished off the twin paradox. Is that really all there is to it? Well, not quite. There's nothing wrong with what we've said so far, but we've left out a lot. There are reasons for the popular confusion.

For one thing, we've been rather unfair to Stella. We've said why she can't simply adopt Terence's viewpoint, but we haven't said how things look from her perspective. It seems passing strange that *Terence* could age several years just because *Stella* engages her thrusters. The Time Gap and Distance Dependence Objections put a sharper edge on this uneasy feeling.

There are versions of the twin paradox where Stella *doesn't* turn on her thrusters and feels no acceleration (the Slingshot variation, where Stella whips round a distant star in free fall, and the Magellan variation, where Stella travels round a cylindrical or spherical universe). These cast doubt on how relevant the acceleration is in the usual version. (We may add FAQ entries for these variations sometime in the future, but at the moment they are left as Exercises for the Reader.)

Finally, what about the Equivalence Principle? Doesn't that say that Stella can *still* claim to be motionless the whole time, but that a huge pseudo-gravitational field just happened to sweep through the universe when she hit her "thrusters on" button? (For that matter, Terence experiences the Earth's gravity, but his frame can be considered to be approximately inertial.) Some people claim that the twin paradox can or even must be resolved *only* by invoking General Relativity (which is built on the Equivalence

Principle). This is not true, but the Equivalence Principle Analysis of the twin paradox does provide some additional analysis of the subject. The EP viewpoint is nearly mandatory for understanding some of the twin paradox variations.

Let's lay out a standard version of the paradox in detail, and settle on some terminology. We'll get rid of Stella's acceleration at the start and end of the trip. Stella flashes past Terence in her spaceship both times, coasting along.

Here's the itinerary *according to Terence:*

Start Event

> Stella flashes past. Clocks are synchronized to 0.

Outbound Leg

> Stella coasts along at (say) nearly 99% light speed. At 99% the time dilation factor is a bit over 7, so let's say the speed is just a shade under 99% and the time dilation factor is 7. Let's say this part of the trip takes 7 years (according to Terence, of course).

Turnaround

> Stella fires her thrusters for, say, 1 day, until she is coasting back towards Earth at nearly 99% light speed. (Stella is the hardy sort.) Some variations on the paradox call for an instantaneous turnaround; we'll call that the *Turnaround Event*.

Inbound Leg

> Stella coasts back for 7 years at 99% light speed.

Return Event

> Stella flashes past Terence in the other direction, and they compare clocks, or grey hairs, or any other sign of elapsed time.

According to Terence, 14 years and a day have elapsed between the Start and Return Events; Stella's clock however reads just a shade over 2 years.

How much over? Well, Terence says the turnaround took a day. Stella's speed was changing throughout the turnaround, and so her time dilation factor was changing, varying between 1 and 7. So Stella's measurement of the turnaround time will be something between 1 day and 1/7 of a day. If you work it out, it turns out to be a bit over 15 hours.

The Twin Paradox: The Doppler Shift Analysis

Let us focus on what Stella and Terence actually **see** with their own eyes. (Just to emphasize that we're talking about direct observation here, I'll put the verb "see" and its brothers in the HTML strong font throughout this section.) To make things interesting, we'll equip them with unbelievably powerful telescopes, so each twin can **watch** the other's clock throughout the trip. If each twin **saw** the other clock run slow throughout the trip, then we *would* have a contradiction. But this is *not* what they **see**.

Just in case it's too hard to read the clock hands through the telescope, we'll add a flash unit to each clock, set to flash once a second. You might guess at first that Terence **sees** Stella's clock flashing once every 7 seconds (with the time dilation factor we've chosen) and vice versa. Not so! On the Outbound Leg, Terence **sees** a flash rate of approximately one flash per 14 seconds; on the Inbound Leg, he **sees** her clock going at about 14 flashes per second. That is, he **sees** it running *fast!* Stella **sees** the same behavior in Terence's clock.

What gives? Well, the section title gave it away: just replace the words "flashes per second" with "cycles per second", and you'll recognize the familiar Doppler shift at work. The regular pulses are redshifted to lower frequencies during the Outbound Leg, and blueshifted to higher frequencies during the Inbound Leg. (I invite you to consider laser-based clocks instead of flash units, for added techno-jazz.)

The Doppler shift factors I gave (1/14 and 14/1) come from the relativistic Doppler formula. The relativistic formula takes into account both the "delay through distance" effect of the non-relativistic formula, and the relativistic time dilation. In other words, Terence *computes* that Stella's clock is really running slow by a factor of about 7 the whole time, but he **sees** it running fast during the Inbound Leg because each flash has a shorter distance to travel. And Stella computes the same for Terence.

All well and good, but this discussion at first just seems to sharpen the paradox! Stella **sees** what Terence **sees**: a slow clock on the Outbound Leg, a fast clock on the Inbound Leg. Whence comes the asymmetry between Stella and Terence?

Answer: in the duration of the Inbound and Outbound Legs, as **seen**. For Stella, each Leg takes about a year. Terence maintains that Stella's turnaround takes place at year 7 at a distance of nearly 7 light-years, so he won't **see** it until nearly year 14. Terence **sees** an Outbound Leg of long duration, and an Inbound Leg of very short duration.

So there's the fundamental asymmetry: the switch from redshift to blueshift occurs at *Stella's* turnaround. Stella **sees** Terence's telescopic image age slowly on her Outbound Leg, but the image more than makes up for its dawdling on the Inbound Leg. Terence **sees** Stella's image off to a slow start too, but here the image's final burst of rapid aging comes too late to win the race.

See the section titled Too Many Analyses for a spacetime diagram of the Doppler Shift Analysis.

The Twin Paradox: The Spacetime Diagram Analysis

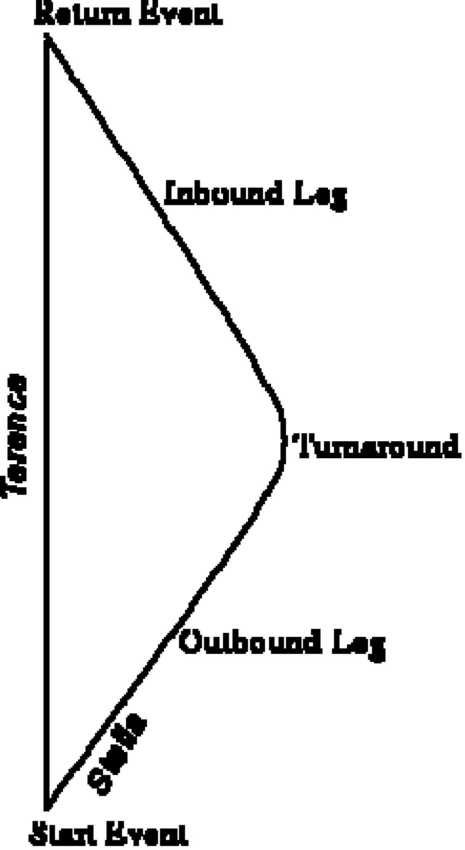


Figure 1: The Twins' Worldlines

This entry borrows heavily from the original FAQ entry for the Twin Paradox, by Kurt Sonnenmoser. However, it has also been extensively modified, so he is not responsible for any sloppiness or infelicities.

Minkowski said "Henceforth Space by itself, and Time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." Minkowski recast Einstein's version of Special Relativity (SR) on a new stage, *Minkowski spacetime*. The Twin Paradox has a very simple resolution in this framework. The crucial concept is the proper time of a moving body.

First chose one specific inertial frame of reference, say the rest frame of the Earth (which we'll pretend is inertial). Once we've chosen a reference frame, we can define co-ordinates (t,x,y,z) for every event that takes place. Pop-science treatments sometimes ask us to imagine an army of observers, all equipped with clocks and rulers, and all at rest with respect to the given reference frame. With their clocks and rulers they can determine *when* and *where* any event takes place--- in other words, its (t,x,y,z) co-ordinates. In a different frame of reference, a different army of observers would determine different co-ordinates for the same event. But we'll stick with one frame throughout this discussion.

The collection of all events in toto, no matter where or when, is called *spacetime*. Traditionally, one plots events in spacetime on a *Minkowski Spacetime Diagram*. That's just a piece of paper (or blackboard!) with the t co-ordinate running vertically upwards, and the x co-ordinate running horizontally. (One just politely ignores the y and z co-ordinates, 4-dimensional paper and blackboards being in short supply at most universities.) Be aware that Terence's lines of simultaneity are horizontal lines on this plot. That is, all events lying on any horizontal line are given the same time co-ordinate by Terence. He regards them as simultaneous.

If we plot all of Terence's and Stella's events, we get their so-called "worldlines". (Miscellaneous trivia: the physicist George Gamow titled his autobiography, "My Worldline".) Terence's and Stella's worldlines are shown in figure 1.

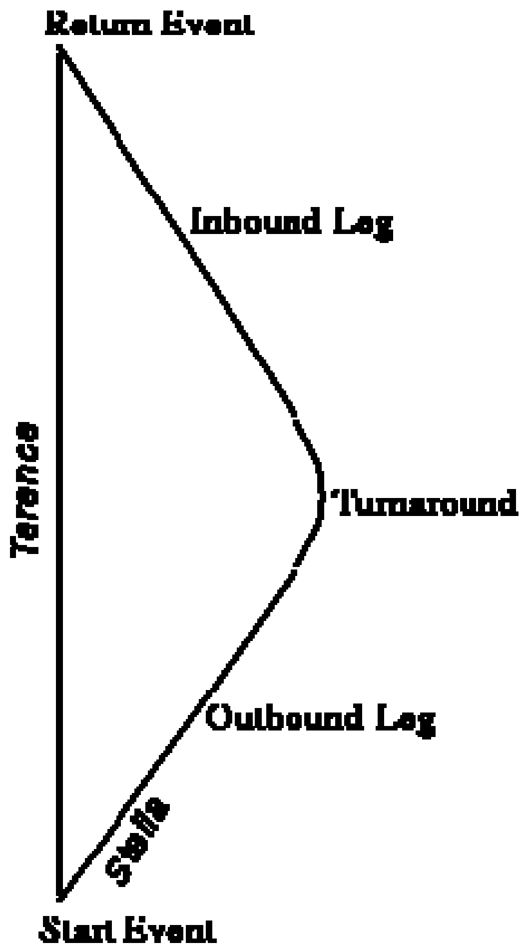


Figure 1: The Twins' Worldlines

Since Terence is at rest in our chosen frame of reference, at all times he will be in same place, say (0,0,0). In other words, the co-ordinates of his events all take this form:

$$(t,0,0,0)$$

But at an arbitrary time *t*, Stella's event co-ordinates will take this form:

$$(t, f(t), g(t), h(t))$$

where *f*(*t*), *g*(*t*), and *h*(*t*) are all functions of *t*, and *t* is (remember) measured by some lowly private in our observer army.

Plotting distance against time is nothing new. Minkowski's new twist was the following formula:

$$d\tau^2 = dt^2 - dx^2 - dy^2 - dz^2$$

Here, *dt*, *dx*, *dy*, and *dz* are all co-ordinate differences between two events that are "near" each other on the Minkowski diagram. (So if (*t*,*x*,*y*,*z*) are the co-ordinates of one event, then (*t*+*dt*,*x*+*dx*,*y*+*dy*,*z*+*dz*) are the co-ordinates of the other.) Time and space are measured in units for which c, the speed of light, equals 1 (e.g. seconds and light seconds). And $d\tau$ is the **proper time** difference, which we define next.

Suppose someone wearing a watch coasts uniformly from event (*t*,*x*,*y*,*z*) to event (*t*+*dt*,*x*+*dx*,*y*+*dy*,*z*+*dz*). The time between these two events, *as measured by that person's watch*, is called the elapsed **proper time** for that person. And according to Minkowski, the proper time is given by $d\tau$ in the formula above.

More generally, suppose someone carrying a high-quality time-piece travels some worldline from event E to event F. "High-quality" here means that acceleration doesn't affect the time-keeping mechanism. A pendulum clock would not be a good choice! A balance-wheel watch might do OK, a tuning-fork mechanism would be still better, and an atomic clock ought to be nearly perfect. How much time elapses according to the time-piece? I.e., what is proper time along that worldline between events E and F? Well, simply integrate $d\tau$:

$$\text{proper time} = \int_E^F d\tau = \int_E^F \sqrt{1 - v(t)^2}\, dt$$

where

$$v(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

is the velocity vector, and $[v(t)]^2$ is the square of its length:

$$[v(t)]^2 = (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2$$

You shouldn't have much difficulty obtaining these formulas from what we've said already.

Our integral for the proper time can be difficult to evaluate in general, but certain special cases are a breeze. Let's take Terence's case first. Remember that his event co-ordinates are always (*t*,0,0,0), so *dx*, *dy*, and *dz* are always 0 for him. So $d\tau$ is just *dt*, and the forbidding integral becomes:

$$\int_E^F dt$$

that is, just the difference in the *t* co-ordinates! In other words, Terence's elapsed proper time is just the elapsed proper time as measured by our army of observers, in the reference frame in which Terence is at rest. It doesn't stretch credulity too far to suppose that Terence *is* one of those observers.

Now how about Stella? For her, *dx*, *dy*, and *dz* are not always all 0. So *dx/dt*, *dy/dt*, and *dz/dt* are also not always all 0, and their *squares* (which appear in the formula for $[v(t)]^2$) are always non-negative, and sometimes positive. So the quantity under the square root is less than or equal to 1, and sometimes strictly less than 1. Conclusion: the value of Stella's integral is less than that of Terence's integral. I.e., her elapsed proper time is less than Terence's. I.e., she ages less.

That's the whole story! We evaluate a path integral along two different paths, and get two different results. Not so different in spirit from picking two points in ordinary Euclidean space, and then evaluating the arc-length integral along two different paths connecting them. It's not just where you're going, it's how you get there. In the words of the unknown poet:

O ye'll tak' the high road
and I'll tak' the low road,
An' I'll be in Scotland afore ye

(But at least our hero and heroine do get to meet again!)

The Twin Paradox: The Equivalence Principle Analysis

Some background

Although most scenarios in Special Relativity are most easily described using inertial frames, there is no reason why these frames absolutely must be used. The Equivalence Principle analysis of the twin paradox simply views the scenario from the frame in which Stella is at rest the whole time. This is not an inertial frame; it's accelerated, so the mathematics is harder. But it can certainly be done. When the mathematics is described fully, what results is that we can treat a uniformly accelerated frame as if it were an inertial frame *with the addition of* a "uniform pseudo gravitational field". By a "pseudo gravitational field", we mean an apparent field (not a real gravitational field) that acts on all objects proportionately to their mass; by "uniform" we mean that the force felt by each object is independent of its position. This is the basic content of the Equivalence Principle.

The Equivalence Principle analysis of the twin paradox does not use any **real** gravity, and so does not use any General Relativity. (General Relativity is the study of *real* gravitational fields, not pseudo ones, so it has nothing to say about the twin paradox.) Nevertheless, what General Relativity does say about real gravitational fields does hold in a restricted sense for pseudo gravitational fields. The one thing we need here is that time runs slower as you descend into the potential well of a pseudo force field. We can use that fact to our advantage when analysing the twin paradox. But it needs to be emphasised that we are *not* using any actual General Relativity here, and no one ever needs to, to analyse the paradox. We are simply grabbing a result about real gravitational fields from General Relativity, because we know (from other work) that it does apply to a pseudo gravitational field.

We begin with a couple of assertions that belong in the realm of General Relativity. (We postpone asking what SR has to say about these assertions.)

- **Free choice of reference frames:** You can describe the physics of a situation in pretty much any reference frame you like, but some frames demand the introduction of force fields that don't show up in other frames. You can call these "pseudo-force fields", or even "pseudo-gravitational fields".
- **Uniform "gravitational" time dilation:** Say you have two identically constructed clocks. One is deep down in a uniform "gravitational" potential well (or "pseudo-potential", if you prefer); the other is higher up. If the two clocks compare rates by sending light signals back and forth, then both will agree that the lower clock runs slower than the higher clock. This can be rephrased as "Time runs slower as you descend into the potential well of a uniform pseudo-force field."

Older books called our first assertion the General Principle of Relativity, but that term has fallen into disuse.

OK, now for the twin paradox

Our usual version, that is. We'll pick a frame of reference in which Stella is at rest the whole time! When she ignites her thrusters for the turnaround, she can assume that a uniform pseudo-gravitational field suddenly permeates the universe; the field exactly cancels the force of her thrusters, so she stays motionless.

(Of course, the frame in which Stella is always at rest in the scenario we have described is not *uniformly* accelerated, so the simpler description of a uniform pseudo gravitational field does not quite apply. But we can consider that description to apply during the period of Stella's turnaround.)

Terence, on the other hand, does not stay motionless in Stella's frame. The field causes him to accelerate, but he feels nothing new since he's in free fall (or rather, Earth as a whole is). There's an enormous potential difference between him and Stella: remember, he's light years from Stella, in a pseudo gravitational field! Stella is far "down" in the potential well; Terence is higher up. It turns out that we can apply the idea of gravitational time dilation here, in which case we conclude that Terence ages years during Stella's turnaround.

Short and sweet, once you have the background! But remember, this is *not* an explanation of the twin paradox. It's simply a description of it in terms of a pseudo gravitational field. The fact that we can do this results from an analysis of accelerated frames within the context of Special Relativity.

As an added bonus, the Equivalence Principle analysis makes short work of Time Gap and Distance Dependence Objections. The Time Gap Objection invites us to consider the limit of an instantaneous turnaround. But in that limit, the pseudo gravitational field becomes infinitely strong, and so does the time dilation. So Terence ages years in an instant---physically unrealistic, but so is instantaneous turnaround.

The Distance Dependence Objection finds it odd that Terence's turnaround ageing should depend on how far he is from Stella when it happens, and not just on Stella's measurement of the turnaround time. No mystery: uniform pseudo-gravitational time dilation depends on the "gravitational" potential difference, which depends on the distance.

You may be bothered by the Big Coincidence: how come the uniform pseudo-gravitational field happens to spring up just as Stella engages her thrusters? You might as well ask children on a merry-go-round why centrifugal force suddenly appears when the carnival operator cranks up the engine. There's a reason why such forces carry the prefix "pseudo".

Real (not pseudo) gravitational time dilation (i.e., fields due to matter) is a different story. These fields are never uniform, and the derivations just mentioned don't work. The essence of Einstein's first insight into General Relativity was this: (a) you can derive time dilation for uniform pseudo-gravitational fields, and (b) the Principle of Equivalence then implies time dilation for gravitational fields. A stunning achievement, but irrelevant to the twin paradox.

You may find pseudo gravitational time dilation a mite too convenient. Where did it come from? Is it just a fudge factor that Einstein introduced to resolve the twin paradox? Not at all. Einstein gave a couple of derivations for it, having nothing to do with the twin paradox. These arguments don't need the Principle of Equivalence. I won't repeat Einstein's arguments (chase down some of the references if you're curious), but I do have a bit more to say about this effect in the section titled Too Many Analyses.

What is General Relativity?

Einstein worked on incorporating gravitation into relativity theory from 1907 to 1915; by 1915, General Relativity had assumed pretty much its modern form. (Mathematicians found some spots to apply polish and gold plating, but the conceptual foundations remain the same.) If you asked him to list the crucial features of General Relativity in 1907, and again in 1915, you'd probably get very different lists. Certainly modern physicists have a different list from Einstein's 1907 list.

Here's one version of Einstein's 1907 list (without worrying too much about the fine points):

General Principle of Relativity

> *All* motion is relative, not just *uniform* motion. You will have to include so-called pseudo forces, however (like centrifugal force or Coriolis force).

Principle of Equivalence

> Gravity is not essentially different from any pseudo-force.

The General Principle of Relativity plays a key role in the Equivalence Principle analysis of the twin paradox. And this principle gave General Relativity its name. Even in 1916, Einstein continued to single out the General Principle of Relativity as a central feature of the new theory. (See for example the first three sections of his 1916 paper, "The Foundation of the General Theory of Relativity", or his popular exposition *Relativity*.)

Here's the modern physicist's list (again, not sweating the fine points):

Spacetime Structure

> Spacetime is a 4-dimensional riemannian manifold. If you want to study it with coordinates, you may use any smooth set of local coordinate systems (also called "charts"). (This free choice is what has become of the General Principle of Relativity.)

Principle of Equivalence

> The metric of spacetime induces a Minkowski metric on the tangent spaces. In other words, to a first-order approximation, a small patch of spacetime looks like a small patch of Minkowski spacetime. Freely falling bodies follow geodesics.

Gravitation = Curvature

> A gravitational field due to matter exhibits itself as curvature in spacetime. In other words, once we subtract off the first-order effects by using a freely falling frame of reference, the remaining second-order effects betray the presence of a true gravitational field.

The third feature finds its precise mathematical expression in the Einstein field equations. This feature looms so large in the final formulation of GR that most physicists reserve the term "gravitational field" for the fields produced by matter. The phrases "flat portion of spacetime", and "spacetime without gravitational fields" are synonymous in modern parlance. "SR" and "flat spacetime" are also synonymous, or nearly so; one can quibble over whether flat spacetime with a non-trivial topology (for example, cylindrical spacetime) counts as SR. Incidentally, the modern usage appeared quite early. Eddington's book *The Mathematical Theory of Relativity* (1922) defines Special Relativity as the theory of flat spacetime.

So modern usage demotes the uniform "gravitational" field back to its old status as a pseudo-field. And the hallmark of a *truly GR* problem (i.e. *not SR*) is that spacetime is *not flat*. By contrast, the free choice of charts---the modern form of the General Principle of Relativity---doesn't pack much of a punch. You can use curvilinear coordinates in flat spacetime. (If you use polar coordinates in plane geometry, you certainly have not suddenly departed the kingdom of Euclid.)

The usual version of the twin paradox qualifies as a pure SR problem by modern standards. Spacetime is ordinary flat Minkowski spacetime. Stella's frame of reference is just a curvilinear coordinate system.

The Spacetime Diagram Analysis is closer to the spirit of GR (vintage 1916) than the Equivalence Principle analysis. Spacetime, geodesics, and the invariant interval: that's the core of General Relativity.

The Twin Paradox: The Time Gap Objection

Try this on for size.

Make the turnaround *instantaneous*. Relativity puts an upper on speed, but no upper limit on acceleration. An instantaneous Turnaround Event is the limiting case of shorter and shorter turnarounds, and so the theory should handle it.

During the Outbound Leg, Terence ages less than two months, according to Stella. (12 Stella-months / time dilation factor of 7.) During the Inbound Leg, Terence also ages less than two months, according to Stella, by the same computation. The Turnaround Event is instantaneous. Total Terence ageing: less than 4 months, it would seem. Yet Terence is supposed to be over 14 years older when Stella returns! Where did the missing time go?

The Doppler Shift Analysis makes short work of this. Stella **sees** (through her telescope) Terence age hardly at all during her Outbound Leg, and nearly the full 14 years during her Inbound Leg. No gap.

Of course, she **calculates** something different, taking into account Doppler shifts and the finite speed of light. These calculations must be based on reference frames. In fact, for Stella to get two months for the Outbound Leg and two months for the Inbound Leg (as in the paragraph above), she has to *switch* inertial reference frames midway through her journey.

Now different inertial reference frames have different notions of simultaneity. The Outbound reference frame says: "*At the same time* that Stella makes her turnaround, Terence's clock reads about two months." The Inbound reference frame says: "*At the same time* that Stella makes her turnaround, Terence's clock reads about 13 years and 10 months." The apparent "gap" is just an accounting error, caused by switching from one frame to another.

The Spacetime Analysis offers the same elucidation, graphically expressed.

Finally, the Equivalence Principle Analysis reply can be found in that entry.

The Twin Paradox: The Distance Dependence Objection

With our "standard example" (see the Introduction), Stella's accounting of Terence's ageing runs like this: one-seventh of a year on the Outbound Leg, one-seventh of a year on the Inbound Leg, and the rest --- 14 years minus two-sevenths --- during the turnaround. You may recall she does the turnaround in a day, according to Terence, or about 15 hours by her own clock. (Let's just say 15, and hang the minutes; the exact figure won't matter.)

Say Stella takes a longer journey, spending 2 years on both the Inbound and Outbound Legs, for a total of 4 years of her time, or 28 years according to Terence. But she still takes the same 15 hours for the turnaround.

So when Stella and Terence have their joyous reunion, Terence is 28 years older (plus a day). This time Stella's accounting of Terence's ageing runs like so: Terence aged two-sevenths years on the Outbound Leg, ditto for the Inbound Leg, and so Terence must have aged over 27 years during the turnaround.

Summing up: according to Stella, Terence ages around 13 years 7 months on the turnaround for the shorter trip, but over 27 years on the turnaround for the longer trip. Yet Stella says the two turnarounds took the same time. And Terence agrees.

The resolutions are similar to those given for the Time Gap Objection. How much Terence ages *during the turnaround* is not something you can directly observe, according to SR. The Doppler Shift Analysis focuses on what Terence and Stella actually **see** through their telescopes, which avoids the difficulty. Stella's accounting is just that: accounting, dependent on particular reference frames, and in particular on switching from one inertial reference frame to another. No wonder that accounting tricks can produce surprising results. See the Time Gap Objection for more details.

Too Many Analyses: a Meta Objection

An old lawyer joke:

"Your Honor, I will show first, that my client never borrowed the Ming vase from the plaintiff; second, that he returned the vase in perfect condition; and third, that the crack was already present when he borrowed it."
Or to quote Shakespeare: "Methinks the lady doth protest too much."

Why so many different analyses? Are the relativists just trying to bamboozle their opponents, like the defence attorney who just has to stir up doubt about the plaintiff's case without giving his own theory of events? Not at all; the physical theory should and does tell a single coherent story here.

Relativity pays the price of permissiveness. It says to us, "Pick whichever frame you like to describe your results. They're all equivalent." No wonder that one analysis ends up looking like three or four.

Most physicists feel that the Spacetime Diagram Analysis is the most fundamental. It does amount to a sort of "Universal Interlingua", enabling one to see how superficially different analyses are really at heart the same.

Figure 1 is the basic spacetime diagram for our hero and heroine. By adding lines one way or another, we will get all the various analyses. (Oh yes: choose units so that c=1 throughout. So light rays plot as 45 degree diagonal lines in all of our diagrams.)

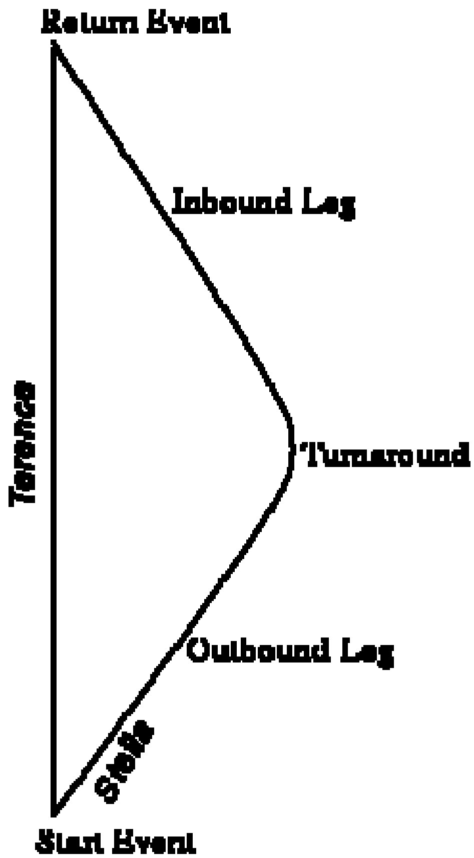


Figure 1: The Twins' Worldlines

Figure 2 is the diagram for the Doppler Shift Analysis. The red lines at 45 degrees are the pulses of light one twin sends to the other. (To reduce clutter, I've made two copies of the diagram. The left one shows Stella's pulses, the right one Terence's.)

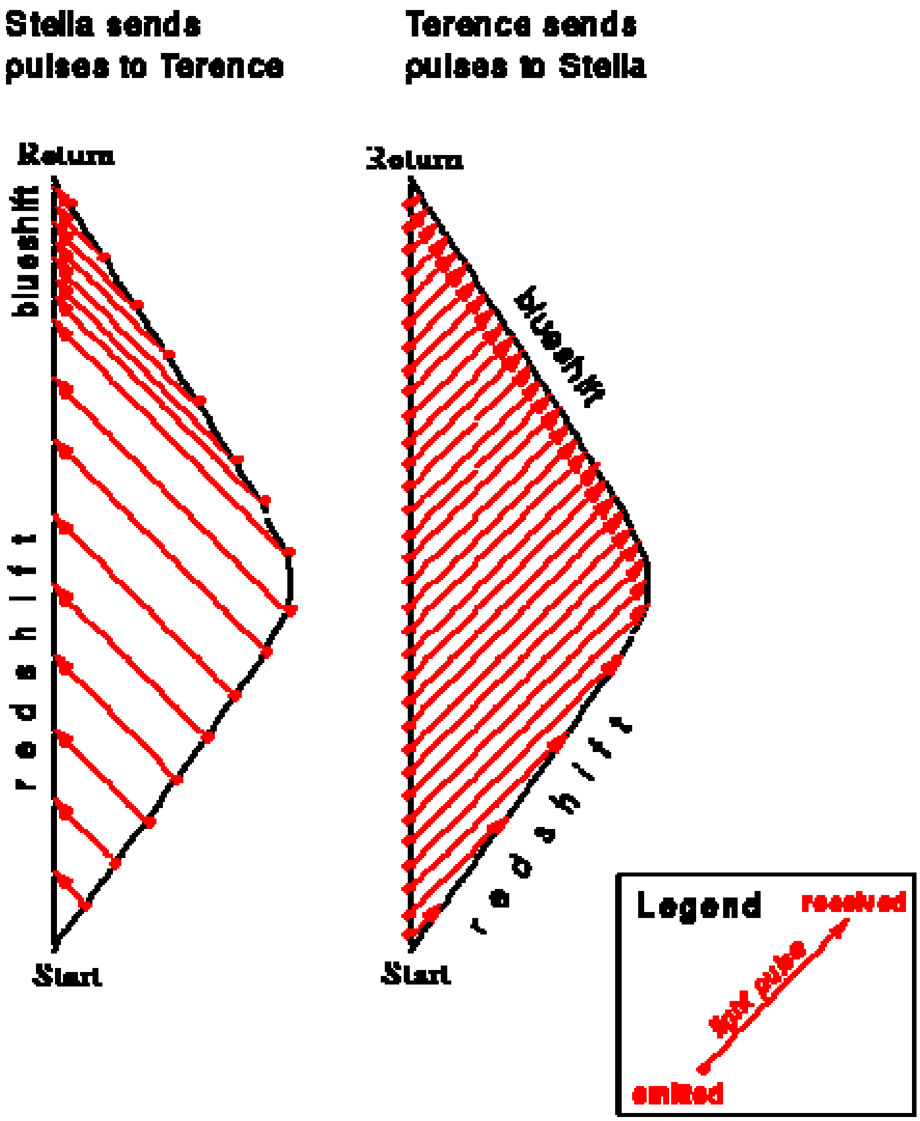


Figure 2: The Doppler Shift Analysis

The time dilation factor in the diagram is two: Terence ages twice as much as Stella. (Notice that Stella has time to send off a mere 16 pulses, while Terence fires off 32.) The emissions are spaced evenly from the viewpoint of the respective *senders*; not so the receptions, which are redshifted or blueshifted according to the relative motion of sender and receiver. All pulses are properly accounted for; check out the Doppler Shift Analysis for full details.

Figure 3, the diagram for the Equivalence Principle Analysis, adds **lines of simultaneity** (in blue) instead of light pulses.

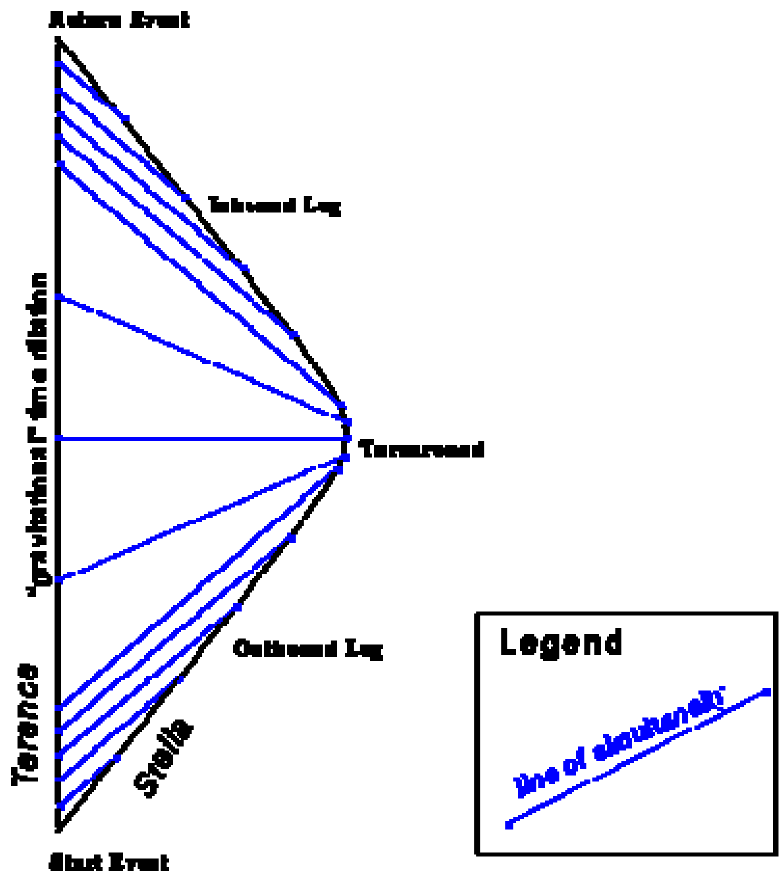


Figure 3

These lines represent collections of events that all happen simultaneously, *according to Stella*. You can see how the lines are closely bunched near Stella, and spread apart near Terence. This is a graphical representation of "pseudo-gravitational" time dilation. From the viewpoint of Stella, her clock is running much *faster* than Terence's during the turnaround.

Modify Figure 3 slightly, and we have a portrayal of the Time Gap Objection (Figure 4).

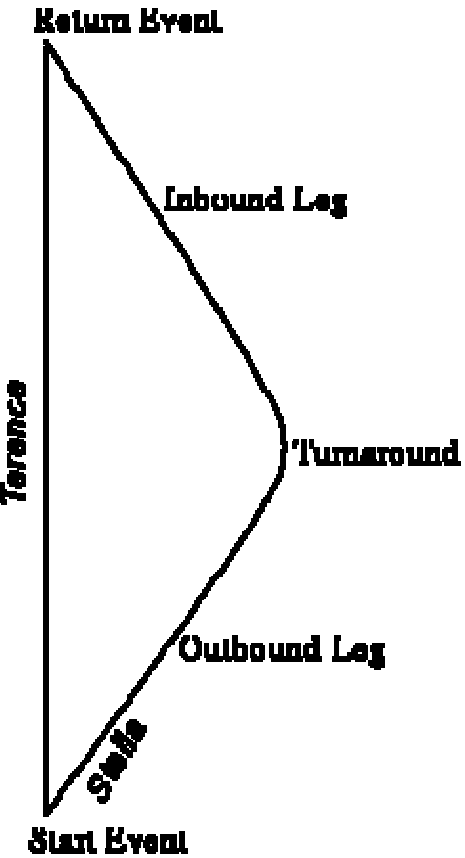


Figure 1: The Twins' Worldlines

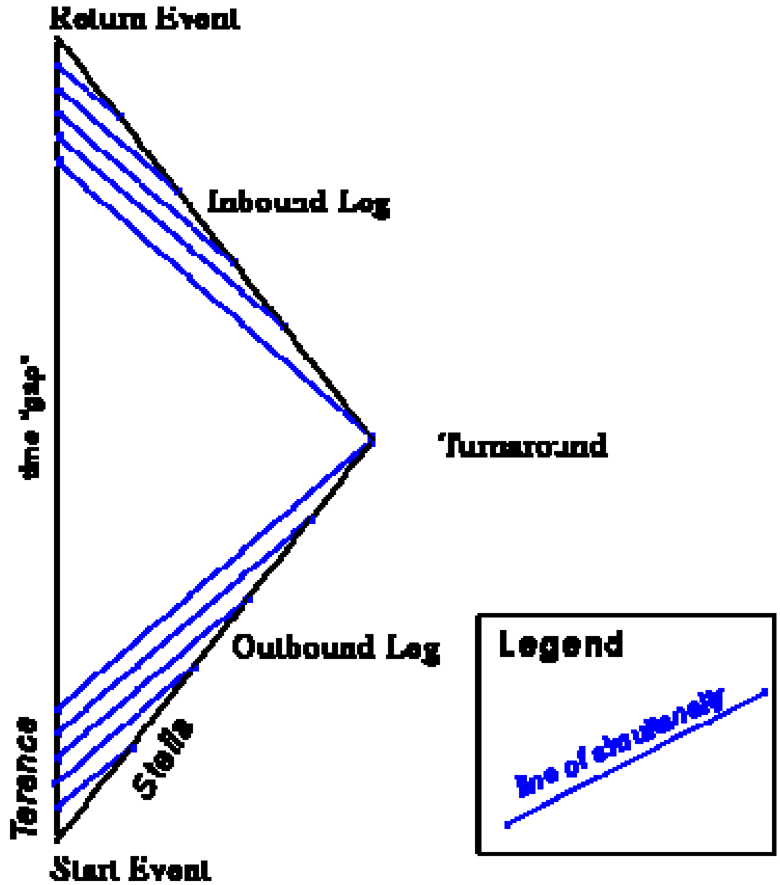


Figure 4: The Time Gap Objection

Here we have let the turnaround become instantaneous. On the Outbound Leg Stella uses one frame of reference, and one notion of simultaneity. On the Inbound Leg she switches to another. The "gap" (the section of Terence's worldline devoid of blue lines) is a consequence of this abrupt switch.

These are just a few of the ways we can decorate our simple diagram with extra lines. In the laissez-faire spirit of General Relativity, we could cover the diagram with almost *any* network of grid lines, and base a description on the resulting coordinate system. (I hasten to add that there are some pitfalls for the unwary: see Section 6.3 of Misner, Thorne, and Wheeler for the fine points.)

On the Twin Paradox in a Universe with a Compact Dimension

Dhruv Bansal, John Laing, Aravindhan Sriharan

April 30, 2009

Abstract

We consider the twin paradox of special relativity in a universe with a compact spatial dimension. Such topology allows two twin observers to remain inertial yet meet periodically. The paradox is resolved by considering the relationship of each twin to a preferred inertial reference frame which exists in such a universe because global Lorentz invariance is broken. The twins can perform "global" experiments to determine their velocities with respect to the preferred reference frame (by sending light signals around the cylinder, for instance). Here we discuss the possibility of doing so with local experiments. Since one spatial dimension is compact, the electrostatic field of a point charge deviates from $1/r^2$. We show that although the functional form of the force law is the same for all inertial observers, as required by local Lorentz invariance, the deviation from $1/r^2$ is observer-dependent. In particular, the preferred observer measures the largest field strength for fixed distance from the charge.

1 Introduction

In the classic presentation of the twin paradox, [1], two observers each witness the other receding at constant velocity and returning at the same velocity at a later time. Each observer will claim he was stationary and, by time dilation, that the *other* should be younger upon meeting. The resolution is that one observer turned around at some point during the journey and, consequently, was not inertial for the entire duration of the trip. This kinematic asymmetry allows both twins to unambiguously determine which of them aged more during the journey: the twin who remained inertial throughout.

In a space-time with one spatial dimension compactified, $S^1 \times \mathbb{R}^{2,1}$, this kinematic solution no longer works. Both twins can remain inertial for the *entire* journey if they confine their motion to the compact dimension (see Fig. 1). In this case, the resolution lies in recognizing that compactifying a spatial dimension breaks global Lorentz invariance, [2]. In particular, there is now a *preferred* inertial reference frame, [2, 3, 4], namely that for which the circle is purely spatial (*i.e.,* the observer whose worldline does not wind around the circle, [5]). The relationship of each observer to this reference frame establishes the asymmetry required to resolve the paradox: the observer in the preferred frame is essentially at rest with respect to the universe and ages more than the moving observer during the journey.

It is well known that observers can determine whether or not they are in the preferred rest frame by sending light beams in opposite directions along the compact dimension, [2, 4]. After waiting for the light beams to traverse the entire compact dimension, only the observer in the preferred frame will receive both signals simultaneously. Moreover, the time interval between the two signals is related to the velocity of the observer relative to the preferred frame.

Such a global experiment is of little practical use if the size of the circle is on the order of cosmological scales since an observer would have to wait about a Hubble time before receiving his signals. Here we present a *local* experiment that either twin can perform to determine his relationship to the preferred frame based on measuring deviations from the $1/r^2$ force law. The electric (or gravitational) field in a universe with a compact dimension is not exactly $1/r^2$ but depends on the size L of the compact dimension because field lines are confined in this direction. Since local Lorentz invariance still holds, the functional form of the field is the same for all inertial observers, but the parameters which appear in the force law, which can be thought of as effective fine-structure (or Newton's) constants, do depend on the observer. This can be understood qualitatively because the size of the compact dimension is not invariant under boosts. A boosted observer sees a larger effective circle (segment, actually) and thus a weaker field. Conversely, an observer in the preferred frame measures the strongest field at fixed distance from the source. Hence by making measurements of the electric field of a point charge stationary in their frame, observers may determine the effective length of the compact dimension in their frame, L_{eff}. Comparing L_{eff} with L, the length of the compact dimension in the preferred frame, precisely specifies the relationship of the observer to the preferred frame and resolves the paradox: the boosted observer ages less during the paradox by a factor $\gamma = L_{eff}/L$.

2 Analysis of the Paradox and a Global Resolution

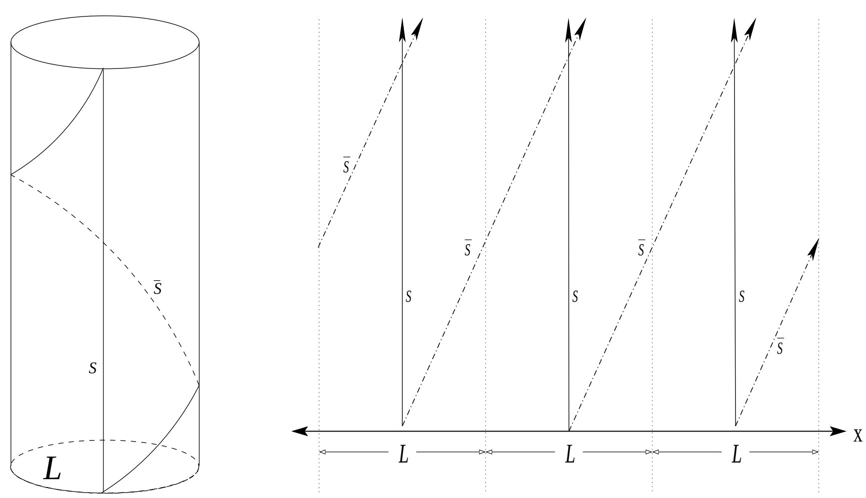
2.1 Description of the Space-time

The manifold we are considering in this problem is the cylinder, $S^1 \times \mathbb{R}^{2,1}$, with the Minkowski metric, $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. It can be constructed from $\mathbb{R}^{2,1}$ by imposing the equivalence relation

$$(t, x, y, z) \sim (t, x + nL, y, z) \tag{1}$$

where L is the circumference of the compact dimension and n is an integer. Each equivalence class of points $[(t, x + nL, y, z)]$ in $\mathbb{R}^{3,1}$ is represented by a single point (t, x, y, z) on the cylinder, chosen such that $0 \leq x < L$.

We thus have two equivalent pictures of the manifold $S^1 \times \mathbb{R}^{2,1}$: the "wrapped" picture, Fig. 1(a), where each point is a unique event, and the "unwrapped" picture in the covering space, Fig. 1(b), where an infinity of points represents the same event. We can consider the latter picture as an infinite sheet of paper which we "wrap" into a cylinder to construct the former picture. It will prove useful to be able to switch back and forth between these two pictures.



(a) (b)

Figure 1: The worldlines of the preferred observer (S) and the non-preferred observer ($\overline{S}$) in the wrapped picture (a) and the unwrapped picture (b).

Lorentz invariance is broken globally since one dimension is compact, which leads to the existence of a preferred rest frame. To see this, consider that the equivalence relation (1) is manifestly dependent on coordinates and so is *itself* defined in a particular frame, call it S. In this frame, a point $p = (t_0, x_0, y_0, z_0)$ in the wrapped picture corresponds to an infinity of points $p_n = (t_0, x_0 + nL, y_0, z_0)$ in the unwrapped picture. These points are all simultaneous in S and they differ from each other only by spatial translations. What about $\overline{S}$, a frame moving with respect to S at some velocity $\beta = v/c$? In this frame, the point p has coordinates $p = (\overline{t_0}, \overline{x_0}, \overline{y_0}, \overline{z_0}) = (\gamma t_0 - \beta\gamma x_0, \gamma x_0 - \beta\gamma t_0, y_0, z_0)$, where $\gamma = (1-\beta^2)^{1/2}$. In the unwrapped picture, this point corresponds to the points $p_n = (\gamma t_0 - \beta\gamma x_0 - \beta\gamma nL, \gamma x_0 + \gamma nL - \beta\gamma t_0, y_0, z_0) = (\overline{t_0} - \beta\gamma nL, \overline{x_0} + \gamma nL, y_0, z_0)$. We see that the equivalence relation (1) in an arbitrary inertial frame $\overline{S}$ becomes

$$(\overline{t}, \overline{x}, \overline{y}, \overline{z}) \sim (\overline{t} - n\beta\gamma L, \overline{x} + n\gamma L, \overline{y}, \overline{z}) \tag{2}$$

Thus, the image points simultaneous in frame $\overline{S}$ are not only translated through space, but through time as well. This is a recognition of the fact that lines of equal time for observers with $\beta \neq 0$ do not close on themselves but spiral around the cylinder. There is only one observer, characterized by $\beta = 0$, whose line of equal time closes on itself, and for whom the identification (2) is a purely spatial one. We refer to the frame of this observer as the *preferred rest frame.*

It should be noted that the effective size of the compact dimension in a frame $\overline{S}$ is γL, as can be seen directly from (2). We thus define the effective length of the compact dimension:

$$L_{eff} \equiv \gamma L \tag{3}$$

The preferred observer measures the smallest value of L_{eff}, namely L.

2.2 Minkowski Diagrams and Transition Functions

The essential problem with the twin paradox in this space-time is that both twins can draw Minkowski diagrams which depict the other twin winding around the circle and coming back. Hence each twin will predict that the other is younger. We can resolve this contradiction by noting that, for the preferred observer, the images of the fundamental domain $(0, L)$ are simply translated spatially by (2). For a non-preferred observer, however, these images are also translated in time. This implies that a diagram like Fig.1(b) is not valid in a non-preferred frame, and that a non-preferred observer cannot naively draw Minkowski diagrams. Instead, the non-preferred observer must take into account certain transition functions.

Because one dimension is compact, observers in our space-time have a problem with multi-valued coordinates. In Sec. 2.1 we glossed over this point and implicitly treated all lengths in the x-dimension modulo L. To be more precise, we should really cover the manifold $S^1 \times \mathbb{R}^{2,1}$ with two single-valued coordinate patches and glue them together with appropriate transition functions.

In some coordinate system S, let patch A cover the entirety of the t, y, and z dimensions and cover an open interval $(0, L_{eff})$ of the compact x-dimension. Likewise, let patch B cover the entirety of the uncompact dimensions and cover an interval $(-\epsilon, \epsilon)$ in the x-dimension. As an analogy, one may think of patch A as a piece of paper wrapped around the cylinder and patch B as a strip of tape applied on the seam of patch A.

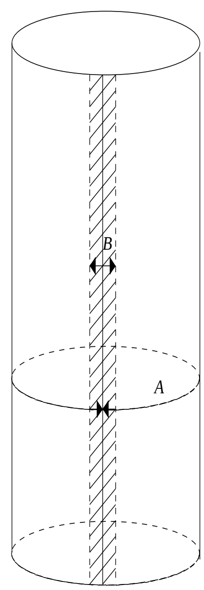


Figure 2: We require two patches, A and B, to cover the manifold $S^1 \times \mathbb{R}^{2,1}$

As we pass from patch A to patch B, we wind around the cylinder or, equivalently, move to another image patch in the unwrapped picture, Fig. 1(b). The index n can thus be thought of as winding number, [5]. If observers keep track of the winding number of light signals, etc., Eq. (2) describes how to relabel paths as their winding number changes. Since a change in winding number corresponds to leaving patch A, crossing through patch B, and returning to patch A, Eq. (2) is recognized to be exactly the transition function we need:

$$f_{\pm}(t,x,y,z) = (t \pm \beta\gamma L, x \mp \gamma L, y, z)\,, \tag{4}$$

where f_+ and f_- are the transition functions used when winding around in the positive and negative x-directions, respectively.

When a given observer attempts to describe the physics within a single patch, say patch A, he must keep track of how to adjust the coordinates he assigns to objects as they exit and then re-enter the patch. For observer S, in the preferred frame, $\beta = 0$, and there is no translation in time as objects wind around the universe. This is why he can naively draw diagrams like Fig. 1(b). For observer $\overline{S}$, however, $\beta \neq 0$, and the transition functions involve translations in time. In this frame, a diagram like Fig. 1(b) would simply be incorrect. Figure 3 properly depicts the situation in both frames using the appropriate transition functions. No contradiction ensues.

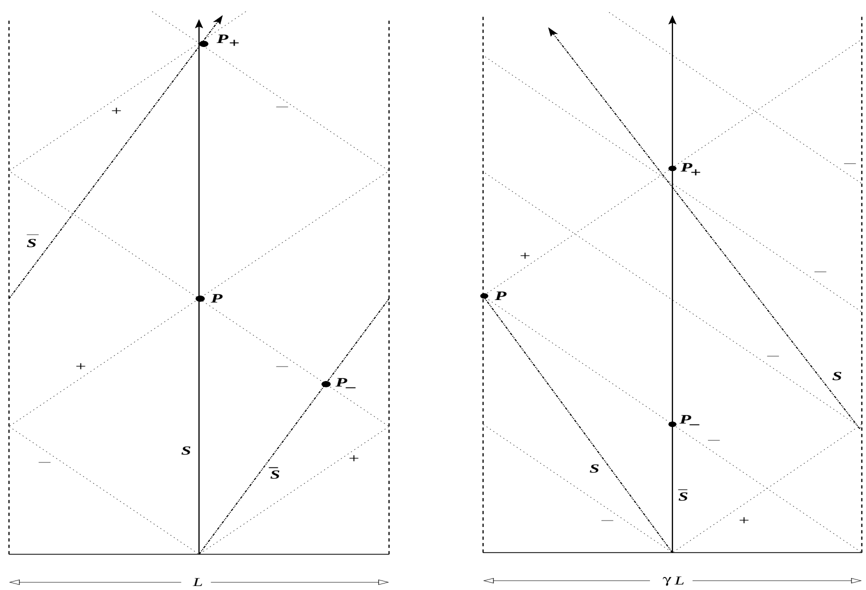
If both twins know their velocity with respect to the preferred frame, then, by Eq. (4), they can find their transition functions and use them to draw correct diagrams. By using the transition functions, both observers in the twin paradox come to the conclusion that the twin in the non-preferred frame ages less during the journey by a factor of γ.

2.3 Einstein Synchronization on the Cylinder

We have thus far used two coordinate patches for the purely mathematical reason of avoiding multi-valued coordinates. The need for multiple patches can, of course, be understood from a physical point of view by considering the synchronization of clocks in this space-time.

The usual method for synchronizing clocks is Einstein synchronization: if an observer is midway between two clocks and receives light signals from each clock with the same reading simultaneously then the two clocks are said to be Einstein synchronized. Usually, Einstein synchronization is a transitive process: if clock A is synchronized with clock B, and clock B is synchronized with clock C, then clock A is synchronized with clock C.

Einstein synchronization immediately fails on a compact dimension because there are two midpoints between any pair of clocks. We can circumvent this problem by choosing a "left-most" and a "right-most" clock. These clocks will demarcate the edges of what will become a coordinate patch. We can synchronize clocks by using the midpoint in between these two boundary clocks, the midpoint in the coordinate patch we are constructing. Transitivity is preserved because we confine all our procedures to this single patch which, without global data, is indistinguishable from an uncompact space.



(a) (b)

Figure 3: Minkowski diagrams depicting the twin paradox from the preferred frame (a) and a non-preferred frame moving with $\beta = 0.5$ (b). The thick lines represent the worldlines of the twins and the dotted lines, labeled by + and −, represent two light signals which wrap around the circle in the positive and negative x-directions, respectively. Note that the preferred observer (S) receives both + and − light signals at the same point (P) while the non-preferred observer ($\overline{S}$) receives them at two different points (P_+ and P_-).

A problem occurs when we let the left-most and right-most clocks approach each other, letting our coordinate patch encircle the entire compact dimension. As soon as they overlap, that is, as soon as the left-most clock and the right-most clock are the *same* clock, we will have constructed a global rest frame and, in a non-preferred frame, this clock will have to read one time to be synchronized with the clock on its left and another time to be synchronized with the clock on its right. This is easily seen by considering lines of equal time in the wrapped picture. For the preferred observer, such lines close on themselves and form circles. For a non-preferred observer, however, they do not close but instead spiral endlessly around the cylinder. The non-preferred observer's coordinate system corresponds to a segment of such a spiral. If this is to span the cylinder, then the segment of the spiral must also span the cylinder. However, if we require each clock to only read one time, this implies that it must be discontinuous at a point. The transition functions (4) are a reflection of this fact. Therefore, while it is possible for the boosted observer to synchronize clocks in this way, evidently this comes at the expense of homogeneity. Indeed, it introduces a special line on the cylinder where time jumps.

In fact, there is a perspicuous analogy between the use of transition functions and patches on this space-time and a more familiar phenomenon: the time zones on the Earth. Imagine a person standing on the equator keeping time by the Sun. In his reference frame, fixed at a point on the Earth's surface, the Sun revolves about the Earth once per day. He attempts to label points on the equator with their distance from him and with a particular time based on the position of the Sun as seen from that point. At a particular moment, let him declare that it is high noon at his own position. Points on the equator east of him will be assigned later times, while points west will be assigned earlier times. As long as his reference frame is local and doesn't span the equator, nothing goes wrong in his scheme. As soon as it does, however, the point diametrically opposite him on the equator demands to be labeled by *two* points in time, one to coincide with the points immediately east of it and another for the points immediately west. His solution is to draw an international date line through that point – a transition function or discontinuity in his coordinate system.

2.4 Global Experiment to Distinguish Twin Observers

To determine his velocity with respect to the preferred frame, an observer can send out light signals in opposite directions along the compact x-dimension, [2]. From Fig. 3 it is clear that the preferred observer, whose transition function does not involve translations in time, will receive the signals at the same time (at the event labeled by P). A non-preferred observer, however, will measure a time-delay in the reception of the two signals (the events P_+ and P_-). A simple calculation yields

$$\beta = \frac{\tau(P_+) - \tau(P_-)}{\tau(P_+) + \tau(P_-)}\,, \tag{5}$$

where $\tau(P)$ is the proper time at which event P occurs. This expression can be used to determine the velocity with respect to the preferred frame. For the preferred observer, one has $\beta = 0$ and indeed $\tau(P_+) = \tau(P_-)$. Once an observer knows his velocity with respect to the preferred frame, he can easily calculate the transition functions (4) and draw appropriate Minkowski diagrams.

3 Electromagnetism and a Local Resolution

The experiment described above would take a prohibitively long time in a universe of any realistic size, as light signals have to encircle the *entire* compact dimension! Furthermore, this global solution does not seem as satisfying as the local solution to the twin paradox in standard space-time $\mathbb{R}^{3,1}$. In the standard space-time, each observer may easily conduct local experiments to determine whether or not he is the accelerated twin – he could hang a pendulum, for example, and watch for any deviations in its path during the journey.

It seems that any local kinematic experiment would not serve to resolve the paradox because there are no *local* kinematic differences between the two observers which might be exploited to distinguish them. The global solution already presented works precisely because it is global - the light beams traverse the entire compact dimension, cross between coordinate patches, and thus force the observers to use transition functions, which encode the relationship between the observer and the preferred rest frame. Here we propose to exploit the local consequences of global phenomena such as electric or gravitational fields. A field permeates all of space and thus "knows" about the global topology. This global knowledge can be extracted by making measurements of the field at a few points.

3.1 Electromagnetism on $S^1 \times \mathbb{R}^{2,1}$

Consider the electromagnetic field of a point charge q at rest at the origin of the preferred rest frame. One expects that the formula for the electromagnetic field of this point charge should deviate from the usual $1/r^2$ since the field lines cannot spread as much in the compact direction. Moreover, such deviations should depend on the size of the circle, L.

To calculate the field, it is easiest to work in the unwrapped picture and consider each image charge as a source for the electromagnetic field at the field point (see Fig. 4). There is no magnetic field, of course, since the point charge and hence all its image charges are at rest in this frame. We find

$$\vec{E}_S(x,y,z) = \frac{q}{4\pi\epsilon_0}\sum_{n=-\infty}^{\infty}\frac{(x+nL)\hat{x}+y\hat{y}+z\hat{z}}{[(x+nL)^2+y^2+z^2]^{\frac{3}{2}}}\,, \tag{6}$$

which depends on L, as expected. It is easy to see that one recovers the usual Coulomb law in the limit $L \to \infty$.

What about the electric field of a point charge stationary in a *non-preferred* frame? Because Lorentz invariance is locally valid in this space-time, the field measured by a non-preferred observer should have the same functional form as Eq. (6) – it can only differ in the values of some parameters. The only parameter to be found in Eq. (6) is the length of the compact dimension, L. Thus we expect L to be replaced with L_{eff}, the effective length of the compact dimension as measured by an observer in a non-preferred frame.

This answer is most easily obtained by noting that a point charge stationary in a non-preferred frame is of course moving at some constant velocity with respect to the preferred observer. From the preferred frame, we can boost directly into the rest frame of the charge and find that we have reproduced the situation we started with prior to deriving Eq. (6): a stationary point charge and an infinite series of image charges, each separated by the effective length of the compact dimension in that frame, L_{eff}. This is illustrated in Fig. 4. Thus, we have

$$\vec{E}_{\overline{S}}(x,y,z) = \frac{q}{4\pi\epsilon_0}\sum_{n=-\infty}^{\infty}\frac{(x+nL_{eff})\hat{x}+y\hat{y}+z\hat{z}}{[(x+nL_{eff})^2+y^2+z^2]^{\frac{3}{2}}} \tag{7}$$

for an arbitrary frame $\overline{S}$. The only change from Eq. (6) is a substitution $L \to L_{eff}$. The field measured by any observer in this universe thus has a dependence on the parameter L_{eff}, the effective length of the universe in the frame of the observer.

A local experiment immediately suggests itself. If we presume observers in this space-time know the value of L, then measuring the electric field of a stationary point charge at a few points is enough to determine L_{eff}, from which one can determine β and resolve the twin paradox.

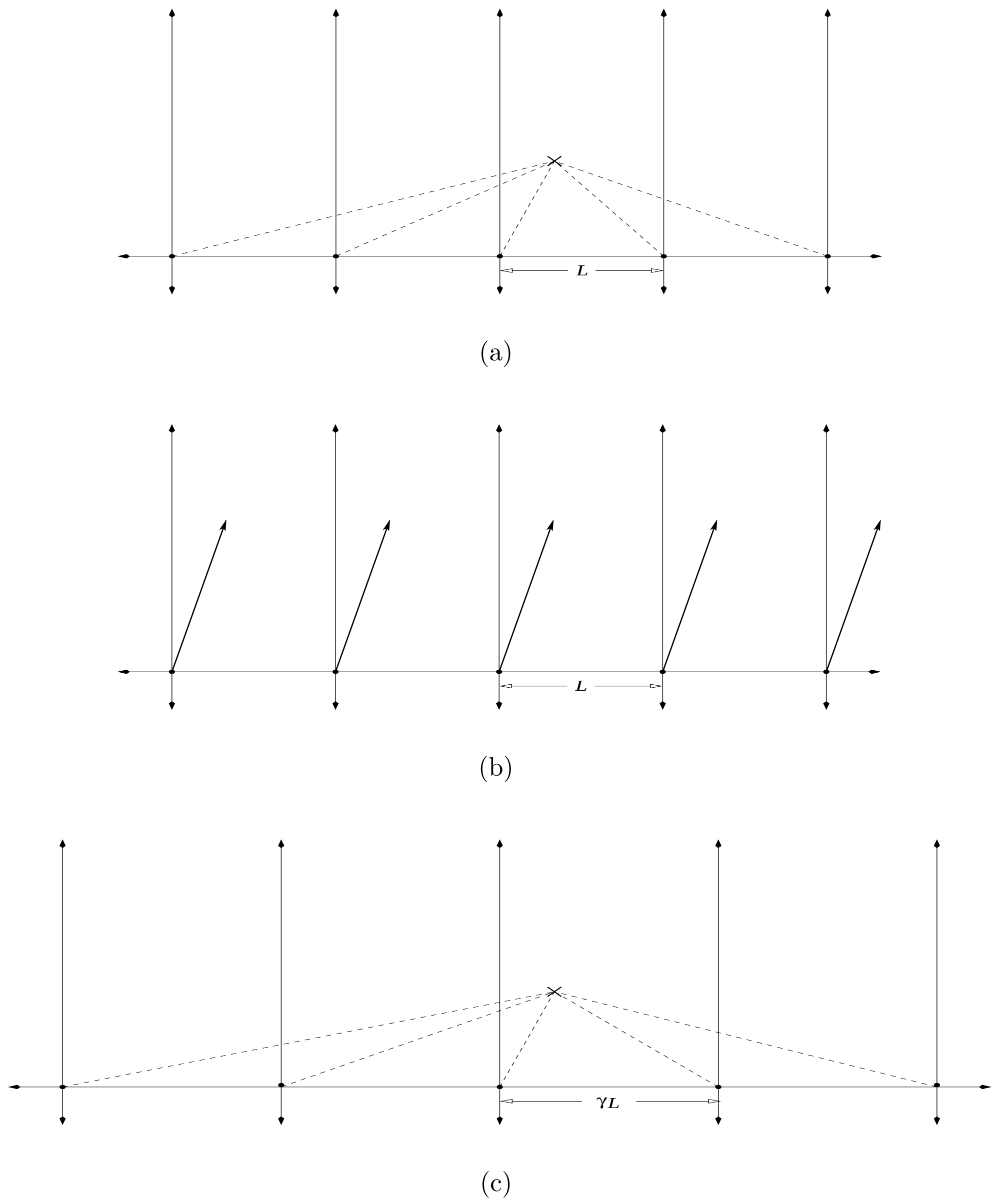


Figure 4: A single image charge stationary at the origin of the preferred frame may be considered as an infinite number of image charges in the unwrapped picture (a). A charge moving at constant velocity in the preferred frame (b) may be considered as a charge stationary at the origin of a boosted, non-preferred frame. In this frame, we may again consider the single charge as an infinite number of image charges (c).

Restricting our attention to points on the x-axis, the infinite sum in Eq. (7) can be written in closed form using residue theorems and then expanded in powers of x/L^2:

$$\begin{aligned}\vec{E}_{\overline{S}}(x,0,0) &= \frac{q}{4\pi\epsilon_0}\sum_{n=-\infty}^{\infty}\frac{\hat{x}}{(x+nL_{eff})^2} \\ &= \frac{q}{4\pi\epsilon_0}\frac{\pi^2}{L_{eff}^2}\csc^2\left(\frac{\pi x}{L_{eff}}\right)\hat{x} \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{x^2}+\frac{\pi^2}{3L_{eff}^2}+\mathcal{O}\left(\frac{x^2}{L_{eff}^4}\right)\right]\hat{x}\,. \end{aligned} \tag{8}$$

It is intriguing to note that the first order correction to the electric field (along the x-axis) in this topology is constant, with the fractional difference from the usual Coulomb field given by

$$\frac{\Delta E}{E}\approx\frac{\pi^2}{3}\left(\frac{x}{L}\right)^2. \tag{9}$$

As expected, the difference increases with decreasing L.

If this experiment is to be practical, however, then the ratio $\Delta E/E$ must not be vanishingly small. The smallest allowed L is $L = 24$ Gpc from cosmic microwave background analysis, [7] (though this figure may require revision, see, [8]). Unfortunately, for any realistic x, this ratio is unmeasurably small. Moreover, it is easily seen that the difference in magnitude between the fields measured in the preferred frame and a non-preferred frame is further suppressed by a factor of β^2 in the non-relativistic limit.

There are two points to be made about the above derivation. First of all, Eq. (8) assumes that the charge has been at rest for sufficiently long so that our expression for the electrostatic field applies. The analysis of a moving charge would require taking into account the self-interactions with radiated photons that circle around the compact dimension and hit the charge back. Secondly, we have completely neglected cosmic expansion and approximated our universe as static. Modeling the paradox on an expanding cylinder (or any compact Friedmann-Robertson-Walker universe) introduces many subtleties, [9, 10].

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